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Partitioning a power set into union-free classes

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Abstract

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Two problems involving union-free colorings of the set of all subsets of an n-set are considered, with bounds obtained for minimum colorings.

1. Introduction

We consider two problems involving 'union-free' colorings of 2^n, the set of subsets of the n-set $[n]$. The first is due to Abbott and Hanson [1]: *for any integer n let $f(n)$ be the minimum number of colors necessary to color 2^n so that each color class is (pairwise) union-free.* That is, no class has three distinct sets A, B, and C such that $A \cup B = C$.

The second function, suggested by Kleitman, is defined in a similar manner: *for any integer n let $g(n)$ be the minimum number of colors necessary to color 2^n so that each color class is (completely) union-free.* That is, for all k no class has distinct sets $A_0, A_1, \ldots, A_k$ such that

$$A_0 = \bigcup_{i=1}^{k} A_i.$$

Here is what we know about f and g.

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Theorem 1.

$$0.35n \approx \frac{\ln 2}{2} n \leqslant f(n) \leqslant \lfloor n/2 \rfloor + 1.$$

Theorem 2.

$$\lfloor n/2 \rfloor + 1 \leqslant g(n) \leqslant n - \mathrm{O}(n^{1/3}).$$

The new results are verified below, the lower bounds of f and g in Section 3 and the upper bound of g in Section 2. We also go over what has been known about bounds for f, giving Abbott and Hanson's upper bound [1], a lovely argument due to Erdős and Shelah [4] yielding a lower bound of about $n/4$, and an improvement due to Aigner and Grieser [2].

2. Upper bounds for $f(n)$ and $g(n)$

Abbott and Hanson [1] observed that there is an upper bound for $f(n)$ given by a partition defined via cardinality.

Consider this partition of 2^n: for $i = 0, 1, \ldots, \lfloor n/2 \rfloor$

$$\mathscr{C}_i \text{ contains all } (2i+1)2^k - 1 \text{ element sets} \quad (\text{for all } k \geqslant 0).$$

This is pairwise union-free and verifies the upper bound for $f(n)$. It is also the correct value for small values of n and we know of no n for which this bound is not equal to $f(n)$.

Turning to $g(n)$, let us take union-free to mean completely so for the rest of this section. To establish the upper bound in Theorem 2 we need some notation.

Let $\mathscr{F}_i$ be the set of i-subsets of $[n]$. The idea is to choose distinct colors c_j for all the sets in $\mathscr{F}_{n-j}$ $(j = 0, \ldots, n-k-1)$ and to color the remaining levels $\mathscr{F}_k, \ldots, \mathscr{F}_0$ with $c_0, \ldots, c_{n-k-1}$ without creating unions and with k as large as possible. We shall show that $k \approx (3n)^{1/3}$ works.

We shall specify that color class $\mathscr{C}_j$ contains some sets from $\mathscr{F}_l$ and $\mathscr{F}_{l-1}$ for some $l \leqslant k$ as well as all of $\mathscr{F}_{n-j}$. This is done so each of the l-sets contains 1, no $(l-1)$-set contains 1, and the union of these l- and $(l-1)$-sets contains at most $n-j-1$ elements. It then follows that $\mathscr{C}_j$ is union-free.

Let $X_1 \mid X_2 \mid \cdots \mid X_{t+1}$ be a partition of $[n]$ into blocks of consecutive integers. For $i \leqslant t$, some of the X_i's may be primed; for instance,

$$\{1,2\}' \mid \{3,4,5\} \mid \{6,7,8\}' \mid \{9,10\}.$$

By i: $X_1 \mid X_2 \mid \cdots \mid X_{t+1}$ denote the family of all i-sets which contain at least one element from every unprimed X_j and have empty intersection with every primed X_j $(j \leqslant t)$. On the last block X_{t+1} there are no restrictions. So, 3: $\{1,2\}' \mid \{3,4,5\} \mid \{6,7,8\}' \mid \{9,10\}$ is the set consisting of 3-sets

$$\begin{array}{ccccc} 3,4,5 & 3,4,9 & 3,4,10 & 3,5,9 & 3,5,10 \\ 4,5,9 & 4,5,10 & 3,9,10 & 4,9,10 & 5,9,10. \end{array}$$

Since the X_i's are sets of consecutive integers, given in order, we may substitute $|X_i|$ for X_i; our example becomes 3: $2', 3, 3', 2$.

Lemma. *Let $l_1, \ldots, l_{i+1}$ and $m_1, \ldots, m_i$ be nonnegative integers with $\sum l_j = \sum m_j = n-1$. Then $A_1^i, \ldots, A_{i+1}^i$ ($B_1^i, \ldots, B_i^i$) defined below form a partition of all the sets in $\mathscr{F}_i$ which do not contain* 1 *(respectively, which contain* 1*):*

$$A_1^i = i\colon\ (l_1+1)',\ n-l_1-1$$
$$A_2^i = i\colon\ 1',\ l_1,\ l_2',\ n-l_1-l_2-1$$
$$\ldots$$
$$A_j^i = i;\ 1',\ l_1,\ \ldots,\ l_{j-1},\ l_j',\ n-\sum_{h=1}^{j} l_h - 1$$
$$\ldots$$
$$A_{i+1}^i = i\colon\ 1',\ l_1,\ l_2,\ \ldots,\ l_i,\ l_{i+1}$$
$$B_1^i = i\colon\ 1,\ m_1',\ n-m_1-1$$
$$B_2^i = i\colon\ 1,\ m_1,\ m_2',\ n-m_1-m_2-1$$
$$\ldots$$
$$B_j^i = i\colon\ 1,\ m_1,\ \ldots,\ m_{j-1},\ m_j',\ n-\sum_{h=1}^{j} m_h - 1$$
$$\ldots$$
$$B_i^i = i\colon\ 1,\ m_1,\ \ldots,\ m_{i-1},\ m_i$$

Proof. Let A be an i-set not containing 1 and let us partition $\{2, 3, \ldots, n\}$ with intervals

$$X_1 = [2, l_1+1], \ldots, X_j = \left[\sum_{h=1}^{j-1} l_h + 2,\ \sum_{h=1}^{j} l_h + 1\right], \ldots,$$
$$X_{i+1} = \left[\sum_{h=1}^{i} l_h + 2,\ n\right].$$

Then with s the maximum index with $A \cap X_j \neq \emptyset$ for $j = 1, \ldots, s$, it is clear that $s \leq i$, $A \in A_{s+1}^i$, and $A \notin A_j^i$ for $j \neq s+1$.

Proceed similarly to show that the B^i's partition the family of i-sets which contain 1; the proof of the Lemma is complete. $\square$

In obtaining the upper bound of $g(n)$ we use these observations about the partitions of the Lemma:

$$\left|\bigcup_{A \in A_j^i} A\right| = n - l_j - 1 \quad (j = 1, \ldots, i), \tag{1}$$

$$\left|\bigcup_{A \in A_{i+1}^i} A\right| = \sum_{j=1}^{i} l_j, \tag{2}$$

$$\left|\bigcup_{B\in B_j^i} B\right| = n - m_j \quad (j=1,\ldots,i-1), \tag{1'}$$

$$\left|\bigcup_{B\in B_i^i} B\right| = \sum_{j=1}^{i-1} m_j + 1. \tag{2'}$$

Call the colors $0, 1, \ldots, n-k-1$ and group them as follows:

$$\begin{array}{ll} & 0, \\ 0: & 1, \ldots, k, \\ 1: & k+1, \ldots, 2k-1, \\ \ldots & \\ i: & ik - \binom{i}{2} + 1, \ldots, (i+1)k - \binom{i+1}{2}, \\ \ldots & \\ k-2: & (k-2)k - \binom{k-2}{2} + 1, \ldots, (k-1)k - \binom{k-1}{2}, \\ k-1: & \text{remaining colors.} \end{array}$$

where we assume that

$$(k-1)k - \binom{k-1}{2} = \frac{k^2+k-2}{2} \leqslant n-k-1.$$

For $1 \leqslant i \leqslant k-1$, we define sequences $l_1^{k-i}, \ldots, l_{k-i+1}^{k-i}$ and $m_1^{k-i}, \ldots, m_{k-i}^{k-i}$ as in the lemma:

$$l_j^{k-i} = (i-1)k - \binom{i-1}{2} + 1 + j \quad (j=1,\ldots,k-i),$$

$$l_{k-i+1}^{k-i} = n - 1 - \sum_{h=1}^{k-i} l_h^{k-i},$$

$$m_j^{k-i} = ik - \binom{i}{2} + 1 + j \quad (j=1,\ldots,k-i-1),$$

$$m_{k-i}^{k-i} = n - 1 - \sum_{h=1}^{k-i-1} m_h^{k-i}.$$

Color $\mathcal{F}_k, \ldots, \mathcal{F}_1$ in this manner. The k-sets in $[2, n]$ are colored 0 and the sets in B_j^k are colored j $(j=1,\ldots,k)$, where the partition of the k-sets containing 1 arises from

$$m_1 = 2,\ m_2 = 3, \ldots, m_{k-1} = k, \quad \text{and} \quad m_k = n - 1 - \sum_{h=1}^{k-1} m_h.$$

The family of 0-colored sets is union-free; so far the j-colored families are as well, this following from (1′) and (2′) for the B_j^k's.

In $\mathcal{F}_{k-i}$ $(i \geq 1)$, color the sets in A_j^{k-i} with $i(i-1)k - \binom{i-1}{2} + j$ and those in B_j^{k-i} with $ik - \binom{i}{2} + j$. To see that these color classes are union-free, consider any color other than 0, say

$$c_{ij} = ik - \binom{i}{2} + j \quad (0 \leq i \leq k-2,\ 1 \leq j \leq k-i).$$

In this class there are the sets of

$$\mathcal{F}_{n-c_{ij}}, \qquad A_j^{k-i-1}, \qquad B_j^{k-i}.$$

Let $j < k-i$. The sets in A_j^{k-i-1} contain a total of $n - l_j^{k-i-1} - 1 = n - c_{ij} - 2$ elements (by (1)), while those in B_j^{k-i} contain $n - m_j^{k-i} = n - c_{ij} - 1$ elements (by (1′)). From the definition of the l's and m's, the elements in these two unions are the same apart from 1, which appears in all the sets in B_j^{k-i} and in none of the sets in A_j^{k-i-1}. Thus, the new use of color c_{ij} results in no forbidden union. Let $j = k-i$. Applying (2) and (2′) the sets in A_{k-i}^{k-i-1} and B_{k-i}^{k-i} contain a total of

$$\left(ik - \binom{i}{2} + 2\right) + \cdots + \left(ik - \binom{i}{2} + (k-i)\right)$$

and

$$\left(ik - \binom{i}{2} + 2\right) + \cdots + \left(ik - \binom{i}{2} + (k-i)\right) + 1$$

elements respectively. Again, these elements are the same, apart from 1, so no union is created if

$$n - (i+1)k - \binom{i+2}{2} \geq 2 + (k-i-1)\left(ik - \binom{i}{2} + 1\right) + \binom{k-i}{2}. \tag{3}$$

By an easy manipulation (3) is equivalent to

$$f(i) = k^2(2i+1) - k(3i^2 + i - 3) + (i^3 - 3i + 2) \leq 2n. \tag{4}$$

By considering the maximum of $f(i)$, it is easily seen that (4) holds if

$$k^3/3(k-i) + k/3(k-i) + (k-i) + 1 \leq n \quad (i = 1, \ldots, k-i). \tag{5}$$

Inequality (5) is valid if

$$k^3/3 + k^2/3 + k + 1 \leq n. \tag{6}$$

Finally, (6) is satisfied with $k \leq cn^{\frac{1}{3}}$, $c = 3^{\frac{1}{3}}$. This completes the proof concerning the upper bound of g.

3. Lower bounds for $f(n)$ and $g(n)$

Concerning $f(n)$, Kleitman [5] showed that for some constant c, no union-free class can contain more than $c(2^n/\sqrt{n})$ subsets of $[n]$. From this Abbott and

Hanson [1] observed that

$$f(n) \geqslant c\sqrt{n}.$$

The lower bound was improved to $\lfloor n/4 \rfloor + 1$ with this argument [4]. For convenience, assume that n is even. Now, consider only intervals $[i, j] = \{i, i+1, \ldots, j\}$ where $i \leqslant n/2 < j$. Let $\mathscr{A}$ be a union-free class of such intervals and define a bipartite graph with vertex sets $\{1, 2, \ldots, n/2\}$ and $\{n/2+1, n/2+2, \ldots, n\}$ with i adjacent to j if and only if the set $[i, j] \in \mathscr{A}$. As $\mathscr{A}$ is union-free, there is no 3-element path in the graph with vertices $i < i' \leqslant n/2 < j' < j$ and edges $j' \sim i \sim j \sim i'$. In particular, the graph has no cycles and, hence, at most $n-1$ edges. Thus, a union-free class has at most $n-1$ such intervals. As there are $n^2/4$ of these intervals, $f(n) \geqslant \lfloor n/4 \rfloor + 1$.

The next contribution bounding $f(n)$ below is a result of Aigner and Grieser [2]: for $n \to \infty$, $0.29n \leqslant f(n)$. This is obtained by investigating hook-free colorings of rectangular arrays.

Here we provide the improvement of the lower bound given in Theorem 1. To begin the proof, let $\mathscr{C}_1, \ldots, \mathscr{C}_s$ be a partition of 2^n such that for distinct A, B, C in any $\mathscr{C}_i$, $A \cup B \neq C$. In showing that $s \geqslant (\ln 2)/2 \approx 0.35n$, we make use of the dual form of the Erdős–Ko–Rado Theorem [3]: for $k \geqslant n/2$ and $\mathscr{A}$ a family of k-subsets of $[n]$ such that $B \cup C \neq [n]$ for all B and C in $\mathscr{A}$, $|\mathscr{A}| \leqslant \binom{n-1}{k}$.

Let $k \geqslant n/2$ and consider all maximal chains in the lattice 2^n. Let y_k^i be the proportion of chains which intersect $\mathscr{C}_i$ in some k-set and let x_k^i be the proportion of chains which intersect $\mathscr{C}_i$ in a k-set and do not intersect $\mathscr{C}_i$ in any set B where $n/2 \leqslant |B| < k$. We claim that for all i and all $k \geqslant n/2$,

$$\sum_i y_k^i = 1, \tag{7}$$

$$\sum_k x_k^i \leqslant 1, \tag{8}$$

$$\frac{n}{2k} y_k^i \leqslant x_k^i. \tag{9}$$

(7) and (8) are easy; here is a proof of (9). Let A be a k-set in $\mathscr{C}_i$. For each maximal chain containing A containing some $B \in \mathscr{C}_i$ such that $B \subset A$ and $|B| \geqslant n/2$, choose the $n/2$-subset C of B on that chain. How many such C's can there be? As $\mathscr{C}_i$ is union-free, the hypotheses of the Erdős–Ko–Rado Theorem apply to the family of C's, showing that there at most

$$\binom{k-1}{\lceil n/2 \rceil}.$$

Therefore, the proportion of $n/2$-subsets of A which are contained in a member of $\mathscr{C}_i$ which is a proper subset of A is at most

$$\frac{\binom{k-1}{\lceil n/2\rceil}}{\binom{k}{\lceil n/2\rceil}} = 1 - \frac{\lceil n/2\rceil}{k} \leqslant 1 - \frac{n}{2k}.$$

Thus, the proportion of maximal chains hitting $\mathscr{C}_i$ at level k and again between levels k and $n/2$ is at most $1 - n/2k$. So the proportion of chains hitting $\mathscr{C}_i$ at level k and not again in a level at or above $n/2$ is at least $(n/2k)y_k^i$, finishing the proof of (9). From (7), (8), and (9) we have

$$\frac{n}{2}\sum_{k=n/2}^{n}\frac{1}{k} = \sum_{k=n/2}^{n}\frac{n}{2k} = \sum_{k=n/2}^{n}\frac{n}{2k}\left(\sum_{i=1}^{s} y_k^i\right)$$

$$= \sum_i \left(\sum_k \frac{n}{2k} y_k^i\right) \leqslant \sum_i \left(\sum_k x_k^i\right) \leqslant \sum_i l = s.$$

Hence, asymptotically,

$$\frac{n}{2}(\ln n - \ln n/2) = \frac{\ln 2}{2} n \leqslant s.$$

Concerning the lower bound of $g(n)$ given in Theorem 2, we show that

$$g(n-2) + 1 \leqslant g(n).$$

As $g(1) = 1$ and $g(2) = 2$, it will follow that $\lfloor n/2 \rfloor + 1 \leqslant g(n)$. Suppose that 2^n has been colored in a completely union-free manner and that $[n]$ has received color a. Then there is some $j \in [n]$ such that no set containing j, except $[n]$, is colored a. Choose any $i \in [n]$ other than j and consider the interval $[\{j\}, [n] - \{i\}]$ in 2^n. This is isomorphic to 2^{n-2} and inherits a union-free coloring without color a. Thus, $g(n-2) + 1 \leqslant g(n)$. $\square$ (Theorem 1 and 2).

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